On the problem on M-hyperquasivarieties

Ewa Graczyńska

[1]

In Memoriam Dietmar Schweigert

Abstract. The aim of this paper is to present a solution of the problem 32 posed by K. Denecke and S.L. Wismath in [11]. It is a continuation of common results of the author and Dietmar Schweigert presented in joint papers [17] – [22].

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1. Notation

Our nomenclature and notation is basically those of G. Birkhoff [7], K. Denecke and S. Wismath [11], G. Grätzer [23], R. McKenzie, G. McNulty and W. Taylor [27]. Some fundamental concepts and properties of algebras and varieties may also be found in [11], therefore we omit them here.

Definition 1.1. A *type* τ of an algebra $\mathbf{A}$ is a function $\tau : I \to \mathbb{N}$ from the indexing set I into the set $\mathbb{N}$ of natural numbers, where $\tau(i) = n_i$ if f_i is an n_i-ary operation. A type τ is finite if the set I is finite.

We deal only with universal algebras of a given *type* $\tau : I \to \mathbb{N}$, where I is a nonempty set and $\mathbb{N}$ denotes the set of all integers.

In that case, for a given $f_i \in F$, $\tau(f_i) = n_i$ is called the *arity* of the operation f_i, $i \in I$ and we will say that f_i is an n_i-ary operation.

Definition 1.2. An *algebra* $\mathbf{A}$ is a pair $\mathbf{A} = (A, F)$, such that: A is a nonempty set; $F = (f_i : i \in I)$ is a family of finitary operations in A.

In the sequel, we shall use the following notation as well:

Definition 1.3. An *algebra* $\mathbf{A}$ is a pair $\mathbf{A} = (A, F^{\mathbf{A}})$, such that: A is a nonempty set; $F = (f_i^{\mathbf{A}} : i \in I)$ is a set of finitary operations in A.

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1.1. Identities and varieties of algebras. By identities of type τ we mean expressions of the form $p \approx q$, where p, q are $n-ary$ polynomial symbols of a given type τ for some $n \in I\!N$.

A *hyperidentity* is formally the same as identity. An identity $x \approx y$ is called *trivial*, where x and y are different variables.

The difference between the concept of an *identity* and *hyperidentity* is in *satisfaction* (see [17], [15] and [45], [11]).

By $Id(\tau)$ we denote the *set of all* identities of type τ. If Σ is a set of identities of type τ, then $E(\Sigma)$ denotes the closure of Σ, i.e. the smallest set of identities of type τ which contains Σ and is closed under the rules of inference (1)-(5) of G. Birkhoff (see G. Birkhoff [7], G. Grätzer [23, p. 170], [27] and W. Taylor [49]).

If V is a variety of algebras of type τ then $Id(V)$ denotes the set of all *identities*, satisfied in V, sometimes called the theory of V. If Σ is a set of identities of type τ, then $Mod(\Sigma)$ denotes the *class of all models*, i.e. the variety of algebras defined by Σ.

For a given algebra $\mathbf{A}$, $Id(\mathbf{A})$ denotes the set of all identities satisfied in $\mathbf{A}$. Shortly speaking, a satisfaction of an identities in an algebra $\mathbf{A}$ means a satisfaction of a pair of terms where the variables are bound by universal quantifiers. A satisfaction of a hyperidentity $p \approx q$ in the algebra $\mathbf{A}$ means its satisfaction as the second-order formula.

Hypersubstitutions of a given type τ of terms were invented by D. Schweigert and the author in [17]. Shortly speaking, they are mappings sending terms to terms by substituting variables by (the same) variables and fundamental terms by terms of the same arities, i.e. $\sigma(x) = x$ for any variable x, and for a given operation symbol f, assume that $\sigma(f(x_0, ..., x_{\tau(f)-1}))$ is a given term of the same arity as f, then σ acts on all terms of a given type τ in an inductive way:

$\sigma(f(p_0, ..., p_{\tau(f)-1})) = \sigma(f(x_0, ..., x_{\tau(f)-1}))(\sigma(p_0), ..., \sigma(p_{\tau(f)-1}))$, for $f \in F$,

(where $f(x_0, ..., x_{\tau(f)-1})$ denotes a *fundamental term*).

A different approaches defined the same concept in [12], [11] and [43].

$H(\tau)$ denotes the set of all hypersubstitutions σ of a given type τ.

$\mathbf{H}(\tau) = (H(\tau), \circ, \sigma_{id})$ denotes the monoid of all hypersubstitutions σ of a given type τ with the operation $\circ$ of composition ant the identity hypersubstitution σ_{id}.

1.2. Quasi-identities and quasivarieties of algebras. We recall the notion invented by A. I. Mal'cev from [30] and [13]:

Definition 1.4. A *quasi-identity* e is an implication of the form:

$$(1.2.1)\ (t_0 \approx s_0) \wedge ... \wedge (t_{n-1} \approx s_{n-1}) \rightarrow (t_n \approx s_n).$$

where $t_i \approx s_i$ are k-ary identities of a given type, for $i = 0, ..., n$.

A *quasi-identity* above is *satisfied in an algebra* $\mathbf{A}$ of a given type if and only if the following implication is satisfied in $\mathbf{A}$: given a sequence $a_1, ..., a_k$ of elements of A. If these elements satisfy the equations $t_i(a_1, ..., a_k) = s_i(a_1, ..., a_k)$ in $\mathbf{A}$, for $i = 0, 1, ..., n-1$, then the equality $t_n(a_1, ..., a_k) = s_n(a_1, ..., a_k)$ is satisfied in $\mathbf{A}$. In that case we write:

$$\mathbf{A} \models (t_0 \approx s_0) \wedge ... \wedge (t_{n-1} \approx s_{n-1}) \rightarrow (t_n \approx s_n).$$

A quasi-identity e is *satisfied in a class* V of algebras of a given type, if and only if it is satisfied in all algebras $\mathbf{A}$ belonging to V.

Following A. I. Mal'cev [30] we consider classes QV of algebras A of a given type τ defined by quasi-identities and call them *quasivarieties*. We use the symbol $QMod(\Sigma)$ for the class QV of algebras of type τ, satisfying a given set Σ of quasi-identities of type τ and call it the *quasivariety axiomatized* by Σ. Σ is then called a *base* of QV.

A *hyperquasi-identity* e (of a type τ) is the same as quasi-identity (of type τ). Sometimes we shall use the notation for *hyperquasi-identity* invented by D. Schweigert in [45]. The difference between quasi-identities and hyperquasi-identities is in satisfaction. Following the ideas of [45], part 5, [11, p. 155] we modified in [18] the definition of the satisfaction of a quasi-identity to the notion of a hypersatisfaction in the following way:

Definition 1.5. A hyperquasi-identity e is *satisfied* (is *hyper-satisfied*, *holds*) in an algebra $\mathbf{A}$ if and only if the following implication is satisfied:
if σ is a hypersubstitution of type τ and the elements $a_1, ..., a_k \in A$ satisfy the equalities $\sigma(t_i)(a_1, ..., a_k) = \sigma(s_i)(a_1, ..., a_k)$ in $\mathbf{A}$, for $i \in \{0, 1, ..., k-1\}$, then the equality $\sigma(t_n)(a_1, ..., a_k) = \sigma(s_n)(a_1, ..., a_k)$ holds in $\mathbf{A}$.

In that case, we write $V \models_H e$.

In other words, a *hyperquasi-identity* is a universally closed Horn $\forall x \forall \sigma$-formulas, where x varies over all sequences of individual variables (occurring in terms of the implication) and σ varies over all hypersubstitutions of a given type. Our modification coincides with Definition 5.1.3 of [45] (see Definition 2.3 of [9]).

Remark. All hyperquasi-identities and hyperidentities are written without quantifiers but they are considered as universally closed Horn $\forall$-formulas (see [30]).

2. Hyperquasi-varieties

A reformulation of the notion of *quasivariety* invented by A. I. Mal'cev in [30, p. 210] to the notion of *hyperquasivariety* of a given type τ was invented by D. Schweigert and the author in [18] in a natural way:

Definition 2.1. A class K of algebras of type τ is called a *hyperquasivariety* if there is a set Σ of hyperquasi-identities of type τ such that K consists exactly of those algebras of type τ that hypersatisfy all the hyperquasi-identities of Σ.

Let us note, that the notion of *hyperqasivariety* coincides with the notion of a *hyperquasi-equational class* invented in [11, p. 155].

3. Hyperquasi-identities

We recall only our definitions of [17] of hyperidentities satisfied in an algebra of a given type and the notion of a *hypervariety*:

Definition 3.1. An algebra $\mathbf{A}$ *satisfies a hyperidentity* $p \approx q$ if for every hypersubstitution $\sigma \in H(\tau)$ the resulting identity $\sigma(p) \approx \sigma(q)$ is satisfied in $\mathbf{A}$. In this case, we write $\mathbf{A} \models_H p \approx q$. A variety V satisfies a hyperidentity $p \approx q$ if every algebra in the variety does. In symbols $V \models_H p \approx q$.

Definition 3.2. A class V of a algebras of a given type is called a *hypervariety* if and only if V is defined by a set Σ of hyperidentities.

In that case we write that $V = HMod(\Sigma)$.

Remark. Some authors avoid to use our concepts since in [51] W. Taylor defined hypervarieties to be classes of varieties of different types satisfying certain sets of equations as identities (see also [38]).

The theorem following was proved in [17]:

Theorem 3.3. *A variety V of type τ is defined by a set of hyperidentities if and only if $V = HSPD(V)$, i.e. V is a variety closed under derived algebras of type τ.*

Let V be a class of algebras of type τ. *Derived algebras* were defined in [10]. *Derived algebras of a given type* τ were defined in [17].

Definition 3.4. Let $\mathbf{A} = (A, F)$ be an algebra in V and σ a hypersubstitution in $H(\tau)$. Then the algebra $\mathbf{B} = (A, (F)^\sigma)$ is a *derived algebra* of $\mathbf{A}$, with the same universe A and the set $(F)^\sigma$ of all derived operations of F by σ. $\mathbf{B}$ is then denoted as $\mathbf{A}^\sigma$.

$D(V)$ denotes the class of all derived algebras of type τ of all algebras of V.

Definition 3.5. A quasivariety V is called *solid* if and only if $D(V) \subseteq V$.

In [18] we presented several theorems of Mal'cev type for solid quasivarieties.

4. M-solid quasivarieties

Let M be a subset of $H(\tau)$ closed under compostion $\circ$ and containing the trivial hypersubstitution i.e. $\mathbf{M}$ is a submonoid $(M, \circ, \sigma_{id})$ of the monoid $(H(\tau), \circ, \sigma_{id})$.

In [20] we reformulated the notion of *hyperquasivariety* of [18] for the case of *M-hyperquasivariety* of a given type in a natural way:

Definition 4.1. A class K of algebras of type τ is called an M-hyperquasivariety if there is a set Σ of M-hyperquasi-identities of type τ such that K consists exactly of those algebras of type τ that M-hypersatisfy all the hyperquasi-identities of Σ.

In [11, p.155] M-hyperquasivarieties were called M-hyperquasi-equational classes.

The following follows from [9] (see also [20]):

Theorem 4.2. *A quasivariety K of algebras given type is an M-hyperquasivariety if and only if it is M-deriverably closed.*

We accept the following definition of [11, p. 155]:

Definition 4.3. Let QV be a quasivariety, then QV is *M-solid* if and only if every M-derived algebra $\mathbf{A}^\sigma$ belongs to QV, for every algebra $\mathbf{A}$ in QV and σ in M, i.e.

$$D_M(QV) \subset QV$$

In [20] we presented some Mal'cev types theorems for *M-hyperquasivarieties*.

5. M-hyperquasi-identities

A suitable generalization of our observations made for the set $H(\tau)$ of all hypersubstitutions was extended in [20] to any subset of $H(\tau)$, closed under composition and containing the trivial hypersubstutuion σ_{id}. This generalization gives rise to so called *M-hypersubstitutions* of a given type.

Definition 5.1. *M-hyperquasi-identity* is formally the same as quasi-identity.

We recall only the definitions of [43] of the fact that a hyperidentity is satisfied in an algebra as an M-hyperidentity of a given type and the notion of M-hypervariety invented in [15]:

Definition 5.2. An algebra $\mathbf{A}$ satisfies a hyperidentity $p \approx q$ as an M-hyperidentity if for every M-hypersubstitution $\sigma \in M$, the resulting identity $\sigma(p) \approx \sigma(q)$ holds in $\mathbf{A}$.

In that case, we write $\mathbf{A} \models_H^M p \approx q$.

A variety V satisfies a hyperidentity $p \approx q$ as M-hyperidentity if every algebra in the variety does. In symbols $V \models_H^M p \approx q$.

Definition 5.3. A class V of algebras of a given type is called an M-hypervariety if and only if V is defined by a set of M-hyperidentities.

In that case we write, that $V = MHMod(\Sigma)$.

Obviously, the definition above generalizes the notion of a hypervariety to an M-hypervariety and a hypersatisfaction to an M-hypersatisfaction. Moreover, every algebra satisfied a set Σ as hyperidentities, satisfies it as a set of of M-hyperidentities. The following was proved in [15]:

Theorem 5.4. *A variety V of type τ is defined by a set Σ of M-hyperidentities if and only if $V = HSPD_M(V)$, i.e. V is a variety closed under M-derived algebras of type τ. Moreover, in this case, the set Σ is then M-hypersatisfied in V and V is the class of all M-hypermodels of Σ, i.e. $V = MHMod(\Sigma)$.*

In order to explain the difference of the notions invented above with the notions of Yu. Movsisyan [32]–[35] we invent the following:

Proposition 5.5. *Let a type τ and the monoid $\mathbf{H}(\tau) = (H(\tau), \circ, \sigma_{id})$ of all hypersubstitutions of type τ be given. Then $M_F(\tau)$ denotes the set of hypersubstitutions σ of type τ, for which $\sigma(f(x_0, ..., x_{\tau(f)-1}))$ is a fundamental term and is not a variable, for every functional symbol $f \in F$. Then the set $M_F(\tau)$ is a submonoid of $H(\tau)$.*

Proof. The proof follows from the fact that the composition of two hypersubstitutions from the set $M_F(\tau)$ is a hypersubstiution in $M_F(\tau)$, which is not a projection (i.e. is not a function determined by a variable). □

Definition 5.6. Every hypersubstitution of type τ from the monoid $M_F(\tau)$ is called a $M_F(\tau)$-*hypersubstitution*. The monoid $\mathbf{M_F}(\tau) = (M_F(\tau), \circ, \sigma_{id})$ is called the monoid of all $M_F(\tau)$-*hypersubstitutions* of type τ.

The following shows the connection of the notions invented in [32]–[35] with the notion of M-hyperidentity of [17]:

Theorem 5.7. *For a given algebra* $\mathbf{A}$ *of type* τ *an identity* $p \approx q$ *is satisfied in* $\mathbf{A}$ *as a hyperidentity in the sense of* [32]*, if and only if it is satisfied in* $\mathbf{A}$ *as an* $M_F(\tau)$*-hyperidentity.*

Definition 5.8. A hyperquasi-identity e is M-*hyper-satisfied* (*holds*) in an algebra $\mathbf{A}$ if and only if the following implication is satisfied:
If σ is a hypersubstitution of M and the elements $a_1, ..., a_n \in A$ satisfy the equalities $\sigma(t_i)(a_1, ..., a_k) = \sigma(s_i)(a_1, ..., a_k)$ in $\mathbf{A}$, for $i = 0, 1, ..., n-1$, then the equality $\sigma(t_n)(a_1, ..., a_k) = \sigma(s_n)(a_1, ..., a_k)$ holds in $\mathbf{A}$.

We say then, that e is an M-hyperquasi-identity of $\mathbf{A}$ and write:

$$\mathbf{A} \models^M_H (t_0 \approx s_0) \wedge ... \wedge (t_{n-1} \approx s_{n-1}) \rightarrow (t_n \approx s_n).$$

A hyperquasi-identity e is M-hyper-satisfied (holds) in a class V if and only if it is M-hypersatisfied in any algebra of V. We write then: $V \models^M_H e$.

By other words, M-hyperquasi-identity is a universally closed Horn $\forall x \forall \sigma$-formulas, where x vary over all sequences of individual variables (occurring in terms of the implication) and σ vary over all hypersubstitutions of M. Our modification coincides with Definition 5.1.3 of [45] and Definition 2.3 of [9].

Remark. All hyperquasi-identities and hyperidentities are written without quantifiers but they are considered as universally closed Horn $\forall$-formulas (see [30]). In case of a trivial monoid M, the notion of M-hypersatisfaction reduces to the notion of classical satisfaction of [7], [10]. If M is the monoid of all hypersubstiutions of a given type τ, then the notion of M-hyperidentity and M-hyperquasi-identity reduces to the hyperidentity and hyperquasi-identity.

Remark. Let us note that in case M is a trivial (i.e. 1-element) monoid of hypersubstitutions of a given type τ, then the satisfaction $\models^M_H$ gives rise to the satisfaction $\models$ and the operator D_M to the identity operator.

In case $M = H(\tau)$ we get the notion of $\models_H$ considered in [18].

6. Examples of M-hyperquasi-identities

6.1. Quasigroups.

Definition 6.1. An algebra $(G, \cdot)$ with a binary operation $\cdot$ is called a *quasigroup*, if for all $a \in G$ the operations $x \cdot a$ and $a \cdot x$ are permutations in G.

This is equivalent to the fact that in the groupoid $\mathbf{G} = (G, \cdot)$ the following two quasi-identities are satisfied:

$$(6.1.1)\ (x \cdot z \approx y \cdot z) \rightarrow x \approx y \text{ and } (6.1.2)\ (z \cdot x \approx z \cdot y) \rightarrow x \approx y.$$

Proposition 6.2. *If a groupoid* $\mathbf{G}$ *satisfies the quasi-identities (6.1.1) and (6.1.2), then these quasi-identities are satisfied in* $\mathbf{G}$ *as* $M_{3,4}$*-hyperquasi-identities, for the monoid* $\mathbf{M_{3,4}} = (M_{3,4}, \circ, \sigma_{id}) = (\{\sigma_3, \sigma_4\}, \circ, \sigma_{id})$*, with:*

$$M_{3,4} = \{\sigma_3, \sigma_4 \in H(2) : \sigma_3(x \cdot y) = x \cdot y, \sigma_4(x \cdot y) = y \cdot x\}.$$

Proof. For $\sigma_3(x \cdot y) = x \cdot y$ and $\sigma_4(x \cdot y) = y \cdot x$ the derived quasi-identities: $\sigma_{3,4}\,(6.1.1, 6.1.2)$ are satisfied in $\mathbf{G}$. Therefore, the quasi-identities (6.1.1) and (6.1.2) are satisfied in $\mathbf{G}$ as $M_{3,4}$-hyperquasi-identities in $\mathbf{G}$.

Note, that the quasi-identities (6.1.1) and (6.1.2) are not satisfied as $M_{1,2}$-hyperquasi-identities for the monoid $M_{1,2}$ generated by the first and the second projections: $\sigma_1, \sigma_2 \in H(2)$. □

6.2. Distributive lattices. The following proposition is an expression of the example 2 presented in [53] in the language of M-hyperidentities:

Proposition 6.3. *In each distributive lattice* $\mathbf{L} = (L, \wedge, \vee)$ *the following identities holds as* $M_F(\tau)$*-hyperidentities, for the monoid* $\mathbf{M_F}(\tau)$ *of all* $M_F(\tau)$*-hypersubstitutions of distributive lattices:*

$$(6.2.1)\ F(x, x) \approx x; \qquad (6.2.2)\ F(x, y) \approx F(y, x);$$
$$(6.2.3)\ F(F(x, y), z) \approx F(x, F(y, z));$$
$$(6.2.4)\ F(x, G(y, z)) \approx G(F(x, y), G(x, z)).$$

Proof. Let us note, that the monoid $M_0(\tau)$ in case of distributive lattices consists of 4 nonequivalent hypersubstitutions (in the sense of [42]) of type (2,2), namely: $\sigma_1(x \wedge y) = x \wedge y$, $\sigma_1(x \vee y) = x \vee y$, $\sigma_2(x \wedge y) = x \vee y$, $\sigma_2(x \vee y) = x \wedge y$, $\sigma_3(x \wedge y) = x \wedge y = \sigma_3(x \vee y)$, $\sigma_4(x \vee y) = x \vee y = \sigma_4(x \vee y)$. The hypersubstitutions of all the identities of (6.2.1) − (6.2.2) are identities satisfied in any distributive lattice $\mathbf{L}$. □

6.3. Boolean Algebras. We express example 1 of [53] in the language of M-hyperidentities:

Proposition 6.4. *The following identity holds as an* $M_F(\tau)$*-hyperidentity in every Boolean algebra:*

$$F(G(x, y)', z)' \approx G(F(x', z)', F(y', z)').$$

6.4. Flat algebras. *Flat algebras* were invented by R. McKenzie. They were considered in [26] as specific 0-*smilattice algebras.*

Let σ be a finite signature containing (among other symbols) a binary symbol $\wedge$ (the meet) and a nullary symnol 0.

Definition 6.5. By a 0-*semilattice* τ-*algebra* we mean an algebra of type τ satisfying the equations

(6.5.1) $x \wedge (y \wedge z) \approx (x \wedge y) \wedge z$;
(6.5.2) $x \wedge y \approx y \wedge x$;
(6.5.3) $x \wedge x \approx x$;
(6.5.4) $f(x_1, ..., x_{i-1}, 0, x_{i+1}, ..., x_n) \approx 0$,
for every n-ary operation f of type τ and every $i \in \{1, ..., n\}$.

Definition 6.6. A *flat algebra* is a 0-semilattice algebra $\mathbf{A}$ such that $a \wedge b = 0$ for all pairs of distinct elements $a, b \in A$.

Consider the monoid $\mathbf{M}_{0,\wedge}(\tau)$ of all prehypersubstitutions of type τ leaving the constant 0 and the operation $\wedge$ unchanged.

Then the following holds:

Theorem 6.7. *The variety of flat algebras is* $\mathbf{M}_{0,\wedge}(\tau)$*-solid.*

Proof. Given a 0-semilattice (flat algebra) $\mathbf{A}$ and a hypersubstitution $\sigma \in M(0, \wedge)$. Then obviously the derived identities of identities (6.5.1) and (6.5.3) remains unchanged and satisfied in $\mathbf{A}$. Consider the derived identity of (6.5.4) by σ, i.e. $\sigma(f(x_1, ..., x_{i-1}, 0, x_{i+1}, ..., x_n)) \approx \sigma(0) \approx 0$, i.e. sis satified in $\mathbf{A}$. Consider the derived identity of (6.5.4) by σ, i.e. $\sigma(f)(x_1, ..., x_{i-1}, 0, x_{i+1}, ..., x_n)) \approx 0$, which is satisfied in $\mathbf{A}$ as $\sigma(f)$ is an n-ary polynomial symbol g of type τ. Moreover, in every 0-semilattice (flat algebra) the following equation holds:

$$(6.5.4^*)\ p(x_1, ..., x_{i-1}, 0, x_{i+1}, ..., x_n) \approx 0,$$

for every nontrivial (i.e. not a variable) term p. We prove this fact by induction on the complexity of the term p, which is not a variable. Assume that the induction hypothesis holds for n-ary polynomials $p_1, ..., p_m$ and let $p = g(p_1, ..., p_m)$ for an m-ary functional symbol g.
Then $g(p_1, ..., p_m)(x_1, ..., x_{i-1}, 0, x_{i+1}, ..., x_n) \approx g(0, ..., 0) \approx 0$. □

Definition 6.8. A 0-semilattice τ-algebra is *compatible* if it satisfies the equation:
(6.5.5) $f(z_1, ..., z_{i-1}, x \wedge y, z_{i+1}, ..., z_n) \approx$
$f(z_1, ..., z_{i-1}, x, z_{i+1}, ..., z_n) \wedge f(z_1, ..., z_{i-1}, y, z_{i+1}, ..., z_n)$,
for every n-ary operation f of type τ and every $i \in \{1, ..., n\}$.

Consider the monoid $\mathbf{M}^*_{0,\wedge}(\tau)$ of all prehypersubstitutions of type τ leaving the constant 0 and the operation $\wedge$ unchenged in such a way, that $\sigma(f)$ is always a functional symbol (of the same arity as f), for every f of type τ. Then the following is obvious:

Theorem 6.9. *The variety of 0-semilattice algebras is* $\mathbf{M}^*_{0,\wedge}(\tau)$*-solid. The variety of compatible flat algebras is* $\mathbf{M}^*_{0,\wedge}(\tau)$ *solid.*

Recall from [26, p. 666] the following definition of *basic x-term* of depth n:

Definition 6.10. The term x is the only *basic x-term* of depth 0. For $n > 0$, *basic x-terms* of depth n are the terms $f(x_1, ..., x_{i-1}, t, x_{i+1}, ..., x_n)$ such that f is an n-ary operation symbol of type τ, $1 \leq i \leq n$, t is a basic x-term of depth $n-1$ and $x_1, ..., x_n$ are variables different of x.

Lemma 6.11. *For every hypersubstitution σ from $\mathbf{M}^*_{0,\wedge}(\tau)$ and every basic x-term $t(x)$ of depth n, the hypersubstitution term $\sigma(t(x))$ is a basic x-term of depht n.*

Proof. The term $\sigma(f(x_1, ...x_{i-1}, t, x_{i+1}, ..., x_n))$ equals to a term of the form:
$g(x_1, ..., x_{i-1}, \sigma(t), x_{i+1}, ..., x_n)$ for some n-ary functional symbol g ot type τ. Therefore the lemma follows by induction on the complexity of a basic x-term $t(x)$. □

Recall from [26, p. 668], that for a finite compatible, flat algebra **A** there exists a finitely q-based (i.e. having a finite base of quasi-identities) quasivariety Q'_A containing **A**. The base constructed contains all identities of the form (6.5.1) – (6.5.5) and quasi-identities constracted by means of basic x- and y-terms of depth $\leq K$, for K being the cardinality of A and the operation $\wedge$. Via Lemma 6.11 we conclude the following slight strengthening of Lemma 3.1 of [26]:

Proposition 6.12. *Q'_A is a finitely q-based $\mathbf{M}_{0,\wedge}(\tau)$-hyperquasivariety containing* **A**.

7. Hyperquasi-equational logic

In this section we present a solution of the following particular case of the Problem 32 [11, p. 291]:

(P.32) *Give the derivation rules for M-hyperquasi-equational logic.*

First, we shall consider the case where the monoid M is trivial, i.e. one-element. In the sequel, **E** denotes the *equational logic*, i.e. the fragment of the first-order logic without relation symbols. The formulas of **E** are all possible identities of a given type τ, the set of axioms Eq of **E** are identities of the form $p \approx p$, and the rules of inferences are the equality rules (atomic formulas are regarded as identities) and the *substitution rule*, i.e. G. Birkhoff's rules (1)–(5) of derivation.
E denotes the set of equality axioms of a given type τ (see [13, p. 33]).

For a set Σ of (hyper)quasi-identities of a given type τ, $HQMod(\Sigma)$ denotes the class of all algebras **A** which hypersatisfy all elements of Σ.

HE denotes the *hyperequational logic*, i.e. the fragment of the second-order logic, extending the equational logic. The formulas and axioms are the same as in **E**. To the inference rules we add the rule (6) of hypersubstitution defined in [17].

Following [13, p. 72] a quasi-identity e is called a *consequence* of the set Σ of quasi-identities if for every algebra **A** of type τ, $\mathbf{A} \models \Sigma$ implies that $\mathbf{A} \models e$. In symbols: $\Sigma \models e$.

We say that an identity e is a *hyperconsequence* of a set of quasi-identities Σ, if for every algebra $\mathbf{A} \in HMod(\Sigma)$, it follows that $\mathbf{A} \models_H e$, i.e. $\mathbf{A} \models_H \Sigma$ implies $\mathbf{A} \models_H e$. In symbols: $\Sigma \models_H e$.

Following [13] we use the following notation:

$$\Delta \to \alpha, \text{ for a set } \Delta = \{p_i \approx q_i : 0 \leq i \leq n-1\} \text{ and } \alpha = p_n \approx q_n$$

instead of the quasi-identity:

$$p_0 \approx q_0 \wedge ... \wedge p_{n-1} \approx q_{n-1} \to p_n \approx q_n.$$

We adopt the convention, that an identity $p \approx q$ may be regarded as a quasi-identity e of the form $\emptyset \to p \approx q$, where $\emptyset$ denotes the empty set.

G. Birkhoff's well known theorem is called *the completeness theorem*:

Theorem 7.1. *An identity e is a consequence of a set Σ of identities if and only if e is derived from Σ in* **E**.

The question naturally arises of when an identity is a consequence of a set of quasiidentities Σ (see [7]). Following [13, p. 72] it is necessary, together with a substitution rule to consider the *modus ponens* rule:

$$\text{(MP)}\ \frac{\alpha, \{\alpha\} \cup \Delta \to \beta}{\Delta \to \beta}.$$

Recall from [13, p. 73], that in the quasi-equational logic $\mathbf{Q}$ (of a given type τ), without relation symbols, the formulas are all possible quasi-identities of a given type τ, the axioms are the *equality axioms* (E.1) – (E.4) and the inference rules are the *substitution rule*, the *cut rule* and the *extension rule*. We list all of them.

Axioms:

(E.1) the reflexivity:

$$p \approx p \to p \approx p,$$

(E.2) the symmetry:

$$p \approx q \to q \approx p,$$

(E.3) the transitivity:

$$(p \approx q) \wedge (q \approx r) \to (p \approx r),$$

or in an equivalent notation:

$$\{p \approx q, q \approx r\} \to p \approx r,$$

(E.4) the compatibility:

$$(t_0 \approx s_0) \wedge ... \wedge (t_{\tau(f)-1} \approx s_{\tau(f)-1}) \to (f(t_0, ..., t_{\tau(f)-1}) \approx f(s_0, ..., s_{\tau(f)-1})),$$

for every operation symbol f of type τ,

or in an equivalent notation:

$$\{t_0 \approx s_0, ..., t_{\tau(f)-1} \approx s_{\tau(f)-1}\} \to (f(t_0, ..., t_{\tau(f)-1}) \approx f(s_0, ..., s_{\tau(f)-1})),$$

for every operation symbol f of type τ.

The inference rules are the following rules:

(7.1) the substitution rule (where δ is a substitution of variables):

$$\frac{\{\gamma_0, ..., \gamma_{n-1}\} \to \beta}{\{\delta(\gamma_0), ..., \delta(\gamma_{n-1})\} \to \delta(\beta)}$$

(7.2) the cut rule:

$$\frac{\Delta \to \alpha, \{\alpha\} \cup \Gamma \to \beta}{\Delta \cup \Gamma \to \beta}$$

(7.3) the extension rule:

$$\frac{\Delta \to \alpha}{\{\beta\} \cup \Delta \to \alpha}.$$

We write $\Sigma \vdash_Q e$ if there exists a derivation of a quasi-identity e from a set Σ of quasi-identities in $\mathbf{Q}$.

The classical result by many authors is the following:

Theorem 7.2. *A quasi-identity e is a consequence of a set Σ of quasi-identities if and only if e is derivable from Σ in $\mathbf{Q}$.*

In symbols: $\Sigma \models_Q e$ if and only if $\Sigma \vdash_Q e$.

We modify *quasi equational logic* $\mathbf{Q}$ to *hyperquasi-equational logic* $\mathbf{HQ}$ by adding a new rule:

(7.4) a hypersubstitution rule (where σ is a hypersubstitution of $H(\tau)$):

$$\frac{(t_0 \approx s_0) \wedge \ldots \wedge (t_{n-1} \approx s_{n-1}) \rightarrow (t_n \approx s_n)}{\sigma(t_0) \approx \sigma(s_0) \wedge \ldots \wedge \sigma(t_{n-1}) \approx \sigma(s_{n-1}) \rightarrow \sigma(t_n) \approx \sigma(s_n)},$$

or in an equivalent notation:

(7.4) a hypersubstitution rule (where σ is a hypersubstitution of $H(\tau)$):

$$\frac{\{\gamma_0, \ldots, \gamma_{n-1}\} \rightarrow \beta}{\{\sigma(\gamma_0), \ldots, \sigma(\gamma_{n-1})\} \rightarrow \sigma(\beta)}.$$

Definition 7.3. By **HQ** we denote the hyperquasi-equational logic, which is an extension of the hyperequational logic **HE** based on the equality axioms E and four rules (7.1) – (7.4) above.

We write $\Sigma \vdash_{HQ} e$ if there exists a derivation of e from Σ in **HQ**.
We write $\Sigma \models_{HQ} e$ if e is a hyperconsequence of Σ, considered as a hyperbase, i.e. if $\mathbf{A} \in HQMod(\Sigma)$, then $\mathbf{A} \models_{HQ} e$.

Definition 7.4. A set Σ of quasi-identities of type τ is called *hyperclosed* if and only if it is closed under the equality axioms, the substitution rule, hypersubstitution rule, the cut rule, the extension rule.

We reformulate the classical results in the following way:

Theorem 7.5. *A set Σ is a set of all (hyper)quasi-identities of type τ, (hyper)satisfied in a class K of algebras of type τ if and only if it is (hyper)closed.*

Proof. If Σ is a set of all hyperquasi-identities hypersatisfied in a class K of algebras of type τ, then it is closed in **Q**, i.e. is closed under the equality axioms and the substitution rule, the cut and the extension rule. In consequence it is also closed under the rules of equational logic. If e is a quasi-identity of Σ, then for every $\sigma \in H(\tau)$, the hypersubstitution $\sigma(e)$ of e by σ is satisfied in K. Therefore Σ is closed under the hypersubstitution rule (7.4). In case if e is an identity of type τ, we conclude that $\sigma(e)$ is satisfied in K for every $\sigma \in H(\tau)$. Therefore Σ is closed under the rule (6) of hypersubstitution, i.e. is hyperclosed.

Assume now, that Σ is hyperclosed. Therefore it is closed. We conclude that Σ is a set of quasi-identities satisfied in a class K of algebras of type τ. As Σ is hyperclosed, therefore for every quasi-identity e of Σ and every $\sigma \in H(\tau)$, the derived quasi-identity $\sigma(e)$ is also satisfied by K, which means that K is a class of algebras of type τ, which hypersatisfies Σ. □

The clue of the next proofs is the following:

Proposition 7.6. *A derivation from Σ in **HQ** means a derivation from* (7.4)(Σ) *in **Q**, i.e. one first need to close the set Σ under the hypersubstitution rule* (7.4) *and then under the equality axioms and other rules. The resulting set will be already closed under all axioms and inference rules of* **HQ**.

More precisely:

Proposition 7.7. *The hypersubstitution rule* (7.4) *commutes with all the axioms and rules of the logic* **HQ**.

Proof. First, we note that the assertion easily holds the equality axioms (E1)–(E3). Moreover, by an easy induction on the complexity of terms, the following generalization of the rule (E.4) is valid in the logic **Q**:

(GE.4) $\{t_0 \approx s_0, ..., t_{\tau(f)-1} \approx s_{\tau(f)-1}\} \to (p(t_0, ..., t_{\tau(f)-1}) \approx p(s_0, ..., s_{\tau(f)-1}))$,
for every term p of type τ.

We prove that if the axiom (E.4) is applied first:

$$\{t_0 \approx s_0, ..., t_{\tau(f)-1} \approx s_{\tau(f)-1}\} \to (f(t_0, ..., t_{\tau(f)-1}) \approx f(s_0, ..., s_{\tau(f)-1})),$$

and then the hypersubstitution rule (7.4) is applied by a hypersubstitution σ:

$$\{\sigma(t_0) \approx \sigma(s_0), ..., \sigma(t_{\tau(f)-1}) \approx \sigma(s_{\tau(f)-1})\} \to (\sigma(f(t_0, ..., t_{\tau(f)-1})) \approx \sigma(f(s_0), ..., (s_{\tau(f)-1}))),$$

then one may apply rule (GE.4) with $p = \sigma(f(x_0, ..., x_n))$, to obtain the resulting quasi-identity: $\{\sigma(t_0) \approx \sigma(s_0), ..., \sigma(t_{\tau(f)-1}) \approx \sigma(s_{\tau(f)-1})\} \to$
$\to (\sigma(f)(\sigma(t_0), ..., \sigma(t_{\tau(f)-1})) \approx \sigma(f)(\sigma(s_0), ..., \sigma(s_{\tau(f)-1})))$.

Now we prove the assertion for the modus ponens rule (MP):

$$\text{(MP)}\ \frac{\alpha, \{\alpha\} \cup \Delta \to \beta}{\Delta \to \beta}.$$

i.e. we will show, that if the (MP) rule is applied first and then the hypersubstitution rule (7.4) is applied to deduce a quasi-identity $e = \sigma(\Delta) \to \sigma(\beta)$, then one may apply the hypersubstitution rule (7.4) first to α and $\alpha \cup \Delta \to \beta$ and then (MP), which leads to the quasi-identity e as well.

Secondly, assume that the substitution rule (7.1) is applied (where δ is a substitution of variables):

$$(7.1)\ \frac{\{\gamma_0, ..., \gamma_{n-1}\} \to \beta}{\{\delta(\gamma_0), ..., \delta(\gamma_{n-1})\} \to \delta(\beta)}$$

and then the hypersubstitution rule (7.4) is applied to get the quasi-identity:

$$(*)\ \{\sigma(\delta(\gamma_0)), ..., \sigma(\delta(\gamma_{n-1}))\} \to \sigma(\delta(\beta))$$

for some hypersubstitution $\sigma \in H(\tau)$ and a substitution δ of variables. Assume that the substitution δ acts on variables $x_0, ..., x_m$ of terms $\gamma_0, ..., \gamma_{n-1}, \beta$ putting: $\delta(x_k) = p_k$, then putting $\delta_1(x_k) = \sigma(p_k)$ on variables of terms $\sigma(p_k)$ of type τ, we get that: $\sigma(\delta_1(\gamma_i)) = \delta_1(\sigma(\gamma_i))$, for $i = 0, ..., n-1$ and $\sigma(\delta_1(\beta)) = \delta_1(\sigma(\beta))$.
We conclude that the quasi-identity (*) equals to the quasi-identity:

$$(*)\ \{\delta_1(\sigma(\gamma_0)), ..., \delta_1(\sigma(\gamma_{n-1}))\} \to \delta_1(\sigma(\beta)),$$

which means that one may apply the hypersubstitution rule (7.4) first and then the substitution rule (7.1) to get the same result.

The proof for the cut rule is similar. Assume that the cut rule (7.2) is applied:

$$(7.2)\ \frac{\Delta \to \alpha, \{\alpha\} \cup \Gamma \to \beta}{\Delta \cup \Gamma \to \beta}$$

and then the hypersubstitution rule (7.4) by a hypersubstitution σ gives rise to the quasi-identity:

$$(**)\ \sigma(\Delta) \cup \sigma(\Gamma) \to \sigma(\beta).$$

Then one may apply the hypersubstitution rule (7.4) by σ to the quasi-identities:

$$\Delta \to \alpha \text{ and } \{\alpha\} \cup \Gamma \to \beta$$

to get the resulting quasi-identity (**) via the cut rule (7.2).

We finalize with the proof of the statement for the extension rule, applying first:

$$(7.3)\ \frac{\Delta \to \alpha}{\{\beta\} \cup \Delta \to \alpha}$$

and assuming that the hypersubstitution rule (7.3) by σ was applied then, leading to the quasi-identity:

$$(^{***})\ \{\sigma(\beta)\} \cup \sigma(\Delta) \to \sigma(\alpha).$$

Then one may apply the hypersubstitution rule (7.4) σ first to the quasi-identity: $\Delta \to \alpha$, to get the resulting quasi-identity (***) as a result of the extension rule (7.3). □

The observation above is a generalization of that which has been already noticed in [14, p. 121], for the fact that derivation rules (1)-(5) of G. Birkhoff and the new rule (6) of hypersubstitution behave similarly, i.e. closing a set Σ of identities under (1)-(6) means, to close Σ under (6) first and then under rules (1)-(5) and we are done.

Therefore, we can say that the hyperquasi-equational logic is the one-step extension of the quasi-equational logic by the hypersubstitution rule (7.4).

We obtain a slight generalization of Corollary 2.2.3 of [13, p. 72]:

Proposition 7.8. *An identity e is a (hyper)consequence of a set Σ of quasi-identities if and only if there is a derivation of e (of $\sigma(e)$, for every $\sigma \in H(\tau)$) from $E \cup \Sigma$ by the substitution rule and modus ponens rule (and the hypersubstitution rule* (7.4)*).*

Proof. The first part of the theorem for **Q** is the classical result (see [13]).

Assume that an identity e is a hyperconsequence of a set Σ, i.e. $\Sigma \models_{HQ} e$. It means, that for every algebra $\mathbf{A}$ if $\mathbf{A} \models_{HQ} \Sigma$, then $\mathbf{A} \models_{HQ} e$. In other words: for every algebra $\mathbf{A}$ if $\mathbf{A} \models_{Q} (7.4)(\Sigma) = \{\sigma(\Sigma) : \sigma \in H(\tau)\}$, then $\mathbf{A} \models_{Q} \sigma(e)$, for every $\sigma \in H(\tau)$. Therefore, we conclude that $(7.4)(\Sigma) \models_{Q} \sigma(e)$, for every $\sigma \in H(\tau)$. Therefore, via Corollary 2.2.3 of [13, p. 72], we conclude, that for every $\sigma \in H(\tau)$ there is a derivation of $\sigma(e)$ from $E \cup \Sigma$ by the substitution rule and the modus ponens rule.

Assume now, that there is a derivation of e from $E \cup \Sigma$ by the substitution, hypersubstitution and modus ponens rule. Then for every $\sigma \in H(\tau)$ there is a derivation of $\sigma(e)$ from $E \cup \Sigma$ by the substitution, hypersubstitution and modus ponens rule. Applying the proposition 7.6, we conclude that there is a derivation of $\sigma(e)$ from the closure $(7.4)(E \cup \Sigma)$ of the set $E \cup \Sigma$ by (7.4), by the substitution and modus ponens rule, for every $\sigma \in H(\tau)$. By Corollary 2.2.3 of [13, p. 72], we conclude that $\sigma(e)$ is a consequence of $(7.4)\Sigma$, for every $\sigma \in H(\tau)$, i.e. $(7.4)\Sigma \models_{Q} \sigma(e)$, for every $\sigma \in H(\tau)$. Therefore $\Sigma \models_{HQ} e$. □

The following is the modification of the classical *completeness theorem* of the logic **Q**:

Theorem 7.9. *A (hyper)quasi-identity e is a (hyper)consequence of a set Σ of (hyper)quasi-identities if and only if it is derivable from Σ in* **(H)Q**.

In symbols: $\Sigma \models_{(H)Q} e$ if and only if $\Sigma \vdash_{(H)Q} e$.

Proof. The part of the theorem for **Q** is the classical result of Selman [47] (see [13, p. 73]).

Assume that $\Sigma \models_{HQ} e$, i.e. if an algebra $\mathbf{A} \in HQMod(\Sigma)$, i.e. if $\mathbf{A} \models_{HQ} \Sigma$, then $\mathbf{A} \models_{HQ} e$. This is equivalent to the implication: if $\mathbf{A} \models_{Q} \sigma(\Sigma)$, for every $\sigma \in H(\tau)$,

then $\mathbf{A} \models_{HQ} e$. Equivalently we write this implication in the following way: if $\mathbf{A} \models_Q$ (7.4)(Σ), then $\mathbf{A} \models_{HQ} e$. From the completeness theorem of G. Birkhoff theorem 2.2.5 [13, p. 73] for the logic $\mathbf{Q}$, we conclude, that if (7.4)(Σ) $\models_Q \sigma(e)$, for every $\sigma \in H(\tau)$, i.e. if $\mathbf{A} \models_Q$ (7.4)(Σ), then $\mathbf{A} \models_Q \sigma(e)$, for every $\sigma \in H(\tau)$. Therefore we conclude the implication:
$\Sigma \models_{HQ} e$, then (7.4)(Σ) $\vdash \sigma(e)$, for every $\sigma \in H(\tau)$. We got: $\Sigma \vdash_{HQ} e$.

Assume now that e is derivable from Σ in $\mathbf{HQ}$, i.e. $\Sigma \vdash_{HQ} e$. By proposition 7.6 we conclude that the quasi-identity e is derivable from (7.4)(Σ) in $\mathbf{Q}$, i.e. (7.4)(Σ) $\vdash_Q e$. Therefore, via completeness theorem for $\mathbf{Q}$, we obtain that (7.4)(Σ) $\models_Q e$, i.e. for every algebra $\mathbf{A}$, such that $\mathbf{A} \models$ (7.4)(Σ) it follows that $\mathbf{A} \models e$. This means, that from $\mathbf{A} \models_{HQ} \Sigma$ it follows that $\mathbf{A} \models e$. The similar argument follows for every derived quasi-identity $\sigma(e)$, of e, for every $\sigma \in H(\tau)$. Namely, if $\Sigma \vdash_{HQ} e$, then for every $\sigma \in H(\tau)$ we conclude, that $\Sigma_{HQ}\sigma(e)$, as if $e_1, ..., e_n$ is a proof of e from Σ in $\mathbf{HQ}$, then: $e_1, \sigma(e_1), ..., \sigma(e_n)$ is a proof of $\sigma(e)$ from Σ in $\mathbf{HQ}$. Therefore we got $\Sigma \vdash_{HQ} \sigma(e)$, for every $\sigma \in H(\tau)$. Finally we conclude that $\Sigma \models_{HQ} e$. $\Box$

8. M-hyperquasi-equational logic

In this section we present a solution of the Problem 32 [11, p. 291]:

(P.32) *Give the derivation rules for M-hyperquasi-equational logic.*

Assume that a monoid $\mathbf{M} = (M, \circ, \sigma_{id})$ of hypersubstitutions of type τ is given.

For a set Σ of (hyper)quasi-identities of a given type τ, $MHQMod(\Sigma)$ denotes the class of all algebras $\mathbf{A}$ which hypersatisfy all elements of Σ. By $\mathbf{MHE}$ we denote the M-*hyperequational logic*, i.e. the fragment of the second-order logic, without relation symbols, extending the equational logic. The formulas and axioms are the same as in $\mathbf{E}$. To the inference rules of $\mathbf{E}$ we add the rule $(6)_M$ of M-hypersubstitution defined by the author in [15].

We modify *quasi equational logic* $\mathbf{Q}$ to M-*hyperquasi-equational logic* $\mathbf{MHQ}$ by adding a new rule:

(8.4) an M-*hypersubstitution rule* (where σ is a hypersubstitution of M):

$$\frac{(t_0 \approx s_0) \wedge \ldots \wedge (t_{n-1} \approx s_{n-1}) \rightarrow (t_n \approx s_n)}{\sigma(t_0) \approx \sigma(s_0) \wedge \ldots \wedge \sigma(t_{n-1}) \approx \sigma(s_{n-1}) \rightarrow \sigma(t_n) \approx \sigma(s_n)},$$

or in an equivalent notation:

(8.4) an M-*hypersubstitution rule* (where σ is a hypersubstitution of M):

$$\frac{\{\gamma_0, \ldots, \gamma_{n-1}\} \rightarrow \beta}{\{\sigma(\gamma_0), \ldots, \sigma(\gamma_{n-1})\} \rightarrow \sigma(\beta)}$$

The rule above generalizes the hypersubstitution rule (7.4) of $\mathbf{HQ}$.

Definition 8.1. By $\mathbf{MHQ}$ we denote the M-hyperquasi-equational logic, which is an extension of M-hyperequational logic $\mathbf{MHE}$, generalizes the logic $\mathbf{HQ}$ and is based on the equality axioms (E.1)–(E.4) and the three inference rules of (7.1) - (7.3) of $\mathbf{Q}$ and the M-hypersubstitution rule (8.4).

We write $\Sigma \vdash^M_{HQ} e$ if there exists a derivation of e from Σ in $\mathbf{MHQ}$.

We write $\Sigma \models^M_{HQ} e$ if e is an M-hyperconsequence of Σ, considered as a hyperbase, i.e. if $\mathbf{A} \in MHQMod(\Sigma)$, then $\mathbf{A} \models^M_{HQ} e$.

Remark. Note, that if M is a trivial monoid, then the logic **MHQ** coincides with the logic **Q**. If $M = H(\tau)$, then the logic **MHQ** coincides with the logic **HQ**.

Definition 8.2. A set of quasi-identities of type τ is called M-*hyperclosed* if and only if it is closed under the equality axioms and the substitution rule, M-hypersubstitution rule, the cut rule and the extension rule.

We generalize the classical results in the following way:

Theorem 8.3. *A set Σ is a set of all (M-hyper)quasi-identities of a class K of algebras of type τ if and only if it is M-hyperclosed.*

Proof. If Σ is a set of (M-hyper)quasi-identities M-hypersatisfied in a class K of algebras of type τ, then it is closed in **Q**, i.e. is closed under the equality axioms and the substitution rule, the cut and the extension rule. In consequence it closed under the rules of equational logic, i.e. G. Birkhoff's rules (1) - (5). If e is a quasi-identity of Σ, then for every $\sigma \in M$, the hypersubstitution $\sigma(e)$ of e by σ is satisfied. Therefore Σ is closed under the M-hypersubstitution rule (8.4). In case if e is an identity, we conclude that $\sigma(e)$ is satisfied for every $\sigma \in M$. Therefore Σ is closed under the rule $(6)_M$ of M-hypersubstitution (which is a particular case of the rule (8.4)), i.e. under **MHE**.

Assume now, that Σ is M-hyperclosed. Therefore it is closed. We conclude, that Σ is a set of quasi-identities satisfied in a class K of algebras of type τ. As Σ is M-hyperclosed, therefore for every quasi-identity e of Σ and every $\sigma \in M$, the derived quasi-identity $\sigma(e)$ is also satisfied by K, which means that K is a class of algebras of type τ, which M-hypersatisfy Σ. $\square$

The clue of the next proofs is the following common generalization of Proposition 7.6:

A derivation from Σ in **MHQ** means a derivation from $(8.4)(\Sigma)$ in **Q**, i.e. one first need to close the set Σ under the hypersubstitution rule (8.4) and then under the equality axioms and the other rules. The resulting set will be already closed under all inference rules of **MHQ**.

More precisely:

Proposition 8.4. *The M-hypersubstitution rule* (8.4) *commutes with all the axioms and rules of the logic* **MHQ**.

In symbols: for a quasi-identity e, the following equivalence holds:

$$\Sigma \vdash^{M}_{HQ} e \text{ if and only if } (8.4)(\Sigma) \vdash_Q e.$$

Proof. By Proposition 7.7 the assertion holds for the set E of equality axioms, as in the rule (8.4) one should consider hypersubstitutions $\sigma \in M$ only. We prove the assertion for the modus ponens rule (MP):

$$(\mathrm{MP})\ \frac{\alpha, \{\alpha \cup \Delta\} \to \beta}{\Delta \to \beta}.$$

i.e. we will show, that if the (MP) rule is applied first and then the M-hypersubstitution rule (8.4) is applied to deduce a quasi-identity $e = \sigma(\Delta) \to \sigma(\beta)$, then one may apply the M-hypersubstitution rule (8.4) first to α and $\alpha \cup \Delta \to \beta$ and then (MP), which leads to the quasi-identity e as well.

Secondly, assume that the substitution rule (7.1) is applied (where δ is a substitution of variables):

$$(7.1)\ \frac{\{\gamma_0,...,\gamma_{n-1}\}\rightarrow\beta}{\{\delta(\gamma_0),...,\delta(\gamma_{n-1})\}\rightarrow\delta(\beta)}$$

and then the hypersubstitution rule (8.4) is applied to get the quasi-identity:

$$(*)\ \{\sigma(\delta(\gamma_0)),...,\sigma(\delta(\gamma_{n-1}))\} \rightarrow \sigma(\delta(\beta))$$

for some M-hypersubstitution $\sigma \in M$ and a substitution δ of variables. Asume that the substitution δ acts on variables $x_0,...,x_m$ of terms $\gamma_0,...,\gamma_{n-1},\beta$ putting: $\delta(x_k) = p_k$, then putting $\delta_1(x_k) = \sigma(p_k)$ on variables of terms $\sigma(p_k)$ of type τ, we get that: $\sigma(\delta_1(\gamma_i)) = \delta_1(\sigma(\gamma_i))$, for $i = 0,...,n-1$ and $\sigma(\delta_1(\beta)) = \delta_1(\sigma(\beta))$.
We conclude that the quasi-identity (*) is equal to the quasi-identity:

$$(*)\ \{\delta_1(\sigma(\gamma_0)),...,\delta_1(\sigma(\gamma_{n-1}))\} \rightarrow \delta_1(\sigma(\beta)),$$

which means that one may apply the M-hypersubstitution rule (8.4) first and then the substitution rule (7.1) to get the same result.

The proof for the cut rule is similar. Assume that the cut rule (7.2) is applied:

$$(7.2)\ \frac{\Delta\rightarrow\alpha,\{\alpha\}\cup\Gamma\rightarrow\beta}{\Delta\cup\Gamma\rightarrow\beta}$$

and then the M-hypersubstitution rule (8.4) by a hypersubstitution $\sigma \in M$ gives rise to the quasi-identity:

$$(**)\ \sigma(\Delta) \cup \sigma(\Gamma) \rightarrow \sigma(\beta).$$

Then one may apply the hypersubstitution rule (8.4) by σ to the quasi-identities:

$$\Delta \rightarrow \alpha \text{ and } \{\alpha\} \cup \Gamma \rightarrow \beta$$

to get the resulting quasi-identity (**) via the cut rule (7.2).

We finalize with the proof of the statement for the extension rule, applied first:

$$(7.3)\ \frac{\Delta\rightarrow\alpha}{\{\beta\}\cup\Delta\rightarrow\alpha}$$

and assuming that the M-hypersubstitution rule (8.4) by σ was applied then, leading to the quasi-identity:

$$(***)\ \{\sigma(\beta)\} \cup \sigma(\Delta) \rightarrow \sigma(\alpha).$$

Then one may apply the M-hypersubstitution rule (8.4) σ first to the quasi-identity: $\Delta \rightarrow \alpha$, to get the resulting quasi-identity (***) as a result of the extension rule (7.3). □

The observation above is a generalization of that the author has already noticed in [14], for the fact that derivation rules (1)-(5) of G. Birkhoff and the new rule $(6)_M$ of hypersubstitution behave similarly, i.e. closing a set Σ of identities under $(1)-(6)_M$ means, to close Σ under $(6)_M$ first and then under rules (1)-(5) and we get that the resulting set is closed under the rules $(1)-(6)_M$.

Therefore, we can say that the M-hyperquasi-equational logic **MHQ** is the one-step extension of the quasi-equational logic **Q** by the M-hypersubstitution rule (8.4).

We obtain a slight generalization of Corollary 2.2.3 of [13, p.72]:

Proposition 8.5. *An identity e is an (M-hyper)consequence of a set of quasi-identities Σ if and only if there is a derivation of e ($\sigma(e)$, for every $\sigma \in M$), from*

$E \cup \Sigma$ by the substitution rule and modus ponens rule (and the M-hypersubstitution rule (8.4)*).*

Proof. Assume that an identity e is an M-hyperconsequence of a set Σ, i.e. $\Sigma \models^M_{HQ} e$. It means, that for every algebra $\mathbf{A}$ if $\mathbf{A} \models^M_{HQ} \Sigma$, then $\mathbf{A} \models^M_{HQ} e$. In other words: for every algebra $\mathbf{A}$ if $\mathbf{A} \models_Q (8.4)(\Sigma)$, then $\mathbf{A} \models_Q \sigma(e)$, for every $\sigma \in M$. Therefore, we conclude that $(8.4)(\Sigma) \models_Q \sigma(e)$, for every $\sigma \in M$. Therefore, via Corollary 2.2.3 of [13, p. 72], we conclude, that for every $\sigma \in M$ there is a derivation of $\sigma(e)$ from $(8.4)(E \cup \Sigma)$ by the substitution rule and the modus ponens rule, i.e. from the set $E \cup \Sigma$ by the substitution rule, the modus ponens rule and the M-hypersubstitution rule.

Assume now, that there is a derivation of e from $E \cup \Sigma$ by substitution, M-hypersubstitution and modus ponens rule. Then for every $\sigma \in M$ there is a derivation of $\sigma(e)$ from $E \cup \Sigma$ by the substitution, M-hypersubstitution and modus ponens rule. Applying Proposition 8.4, we conclude that there is a derivation of $\sigma(e)$ from the closure $(8.4)(E \cup \Sigma)$ of the set $E \cup \Sigma$ by (8.4), by the substitution and modus ponens rule, for every $\sigma \in M$. By Corollary 2.2.3 of [13, p. 72], we conclude that $\sigma(e)$ is a consequence of $(8.4)\Sigma$, for every $\sigma \in M$, i.e. $(8.4)\Sigma \models_Q \sigma(e)$, for every $\sigma \in M$. Therefore $\Sigma \models^M_{HQ} e$. $\square$

Remark. Note, that in case of a trivial M, Proposition 8.5 is nothing else but Corollary 2.2.3 of [13, p. 72].

Theorem 8.6. *A (hyper)quasi-identity e is an M-hyperconsequence of a set Σ of (hyper)quasi-identities if and only if it is derivable from Σ in* **MHQ**.

In symbols: $\Sigma \models^M_{HQ} e$ if and only if $\Sigma \vdash^M_{HQ} e$.

Proof. Assume that $\Sigma \models^M_{HQ} e$, i.e. if an algebra $\mathbf{A} \in MHQMod(\Sigma)$, i.e. if $\mathbf{A} \models^M_{HQ} \Sigma$, then $\mathbf{A} \models^M_{HQ} e$. This is equivalent to the implication: if $\mathbf{A} \models_Q \sigma(\Sigma)$, for every $\sigma \in M$, then $\mathbf{A} \models^M_{HQ} e$. Equivalently we write this implication in the following way: if $\mathbf{A} \models_Q (8.4)(\Sigma)$, then $\mathbf{A} \models^M_{HQ} e$. From the completeness theorem of G. Birkhoff Theorem 2.2.5 [13, p. 73] for the logic $\mathbf{Q}$, we conclude, that if $(8.4)(\Sigma) \models_Q \sigma(e)$, for every $\sigma \in M$, i.e. if $\mathbf{A} \models_Q (8.4)(\Sigma)$, then $\mathbf{A} \models_Q \sigma(e)$, for every $\sigma \in M$, i.e. $(8.4)(\Sigma) \models_Q \sigma(e)$, for every $\sigma \in M$.
Therefore we conclude the implication:
$\Sigma \models^M_{HQ} e$, then $(8.4)(\Sigma) \vdash \sigma(e)$, for every $\sigma \in M$. We got: $\Sigma \vdash^M_{HQ} e$.

Assume now that e is derivable from Σ in **MHQ**, i.e. $\Sigma \vdash^M_{HQ} e$. By proposition 8.4 we conclude that the quasi-identity e is derivable from $(8.4)(\Sigma)$ in $\mathbf{Q}$, i.e. $(8.4)(\Sigma) \vdash_Q e$. Therefore, via completeness theorem for $\mathbf{Q}$, we obtain that $(8.4)(\Sigma) \models_Q e$, i.e. for every algebra $\mathbf{A}$, such that $\mathbf{A} \models (8.4)(\Sigma)$ it follows that $\mathbf{A} \models e$. This means, that from $\mathbf{A} \models^M_{HQ} \Sigma$ it follows that $\mathbf{A} \models e$. The similar argument follows for every derived quasi-identity $\sigma(e)$, for every $\sigma \in M$. Namely, if $\Sigma \vdash^M_{HQ} e$, then for every $\sigma \in M$ we conclude, that $\Sigma \vdash^M_{HQ} \sigma(e)$, as if $e_1, ..., e_n$ is a proof of e from Σ in **MHQ**, then: $e_1, \sigma(e_1), ..., \sigma(e_n)$ is a proof of $\sigma(e)$ from Σ in **MHQ**. Therefore we got, that $\Sigma \vdash \sigma(e)$. Finally we conclude that $\Sigma \models^M_{HQ} e$. $\square$

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Opole University of Technology, Institute of Mathematics
ul. Luboszycka 3, 45-036 Opole, Poland
E-mail address: `egracz@po.opole.pl` `http://www.egracz.po.opole.pl/`